NORMAL GENERATION AND CLIFFORD INDEX

YOUNGOOK CHOI[1], SEONJA KIM[2], AND YOUNG ROCK KIM[3]

Abstract. Let C be a smooth curve of genus $g \geq 4$ and Clifford index c. In this paper, we prove that if C is neither hyperelliptic nor bielliptic with $g \geq 2c+5$ and $\mathcal{M}$ computes the Clifford index of C, then either $\deg\mathcal{M} \leq \frac{3c}{2}+3$ or $|\mathcal{M}| = |g^1_{c+2} + h^1_{c+2}|$ and $g = 2c+5$. This strengthens the Coppens and Martens' theorem ([CM91], Corollary 3.2.5). Furthermore, for the latter case (1) $\mathcal{M}$ is half-canonical unless C is a $\frac{c+2}{2}$-fold covering of an elliptic curve, (2) $\mathcal{M}(F)$ fails to be normally generated with $\text{Cliff}(\mathcal{M}(F)) = c$, $h^1(\mathcal{M}(F)) = 2$ for $F \in g^1_{c+2}$. Such pairs $(C, \mathcal{M})$ can be found on a $K3$-surface whose Picard group is generated by a hyperplane section in $\mathbb{P}^r$. For such a $(C, \mathcal{M})$ on a K3-surface, $\mathcal{M}$ is normally generated while $\mathcal{M}(F)$ fails to be normally generated with $\text{Cliff}(\mathcal{M}) = \text{Cliff}(\mathcal{M}(F)) = c$.

1. Introduction

Let C be an irreducible projective curve over an algebraically closed field of characteristic zero. A smooth curve C in $\mathbb{P}^r$ is said to be projectively normal if the natural morphisms $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(C, \mathcal{O}_C(m))$ are surjective for every nonnegative integer m. A line bundle $\mathcal{L}$ on a smooth curve C is said to be normally generated if $\mathcal{L}$ is very ample and C has a projectively normal embedding via its associated morphism $\phi_{\mathcal{L}} : C \to \mathbb{P}(H^0(\mathcal{L}))$.

Green and Lazarsfeld gave a sufficient condition for a line bundle to be normally generated as follows ([GL86], Theorem 1): If $\mathcal{L}$ is a very

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ample line bundle on C with $\deg\mathcal{L} \geq 2g+1-2h^1(\mathcal{L})-\mathrm{Cliff}(C)$, then $\mathcal{L}$ is normally generated. Note that the condition $\deg\mathcal{L} \geq 2g+1-2h^1(\mathcal{L})-\mathrm{Cliff}(C)$ is equivalent to $\mathrm{Cliff}(\mathcal{L}) < \mathrm{Cliff}(C)$, whence $h^1(\mathcal{L}) \leq 1$. A very ample line bundle $\mathcal{L}$ on a smooth curve C is said to be *extremal* if $\mathrm{Cliff}(\mathcal{L}) = \mathrm{Cliff}(C)$ and $\mathcal{L}$ fails to be normally generated.

In this paper, we show that there are extremal line bundles $\mathcal{L}$ on smooth curves with $h^1(\mathcal{L}) = 2$. The existence of an extremal line bundle $\mathcal{L}$ with $h^1(\mathcal{L}) \leq 1$ can be found in [GL86], [Ko02];

Theorem 1.1 (Green-Lazarsfeld, [GL86]). *Let C be a smooth curve of genus g and Clifford index c. If $g > \max\{\binom{c+3}{2}, 10c+6\}$, then:*
(a) C always carries an extremal line bundle $\mathcal{L}$ with $h^1(\mathcal{L}) = 0$, but never one with $h^1(\mathcal{L}) \geq 2$;
(b) C carries an extremal line bundle $\mathcal{L}$ with $h^1(\mathcal{L}) = 1$ if and only if $c = 2f \geq 4$ is even, and C is a two-sheeted branched covering $\pi : C \to C' \subset \mathbb{P}^2$ of a smooth plane curve C' of degree $f+2$.

Thus all the extremal line bundle $\mathcal{L}$ with $h^1(\mathcal{L}) \geq 2$ could be found in case $\mathrm{Cliff}(C)$ is not so small compared to the genus g. By the way it is well known that $\mathrm{Cliff}(C) \leq [(g-1)/2]$ for any smooth curve and $\mathrm{Cliff}(C) = [(g-1)/2]$ if C is a general curve [Me60]. To find an extremal line bundle $\mathcal{L}$ with $h^1(\mathcal{L}) \geq 2$, we investigate the property of a line bundle computing the Clifford index of C and the normal generation of a line bundle on C. We prove the following theorem which supplies a tool constructing various non-normally generated line bundles on a curve.

Theorem 1.2. *Let $\mathcal{M}$ be a birationally very ample line bundle on a smooth curve C, $\mathcal{F}$ be globally generated with $h^0(\mathcal{F}) = 2$ and $F \in |\mathcal{F}|$. Assume that*

(1) $h^1(\mathcal{M}^2(F)) = 0$;
(2) $h^1(\mathcal{M}^2) + 1 \leq h^1(\mathcal{M}^2(-F))$;
(3) $H^0(\mathcal{M}) \otimes H^0(F) \to H^0(\mathcal{M}(F))$ *is surjective.*

Then $\mathcal{L}(:= \mathcal{M}(F))$ fails to be normally generated, especially the 2-normality of $\phi_{\mathcal{L}}(C)$ does not hold.

Note that hypotheses except (3) in the above theorem could be easily checked and possibly hold. For example, if $\mathcal{M}^2$ is special then hypothesis (2) naturally holds by $h^0(\mathcal{F}) = 2,\ h^1(\mathcal{M}^2(F)) = 0$ and the base point free pencil trick. For condition (3) we give a geometric criterion. The line bundle $\mathcal{L}$ becomes extremal if $\mathcal{L}$ computes the Clifford index of C. Generally it is hard to determine whether $\mathcal{L}$ computes the Clifford index of C or not.

The following theorem makes it possible to find out line bundles $\mathcal{M}$ and $\mathcal{F}$ satisfying the assumptions of Theorem 1.2 such that $\mathcal{M}(F)$ is an extremal line bundle with $h^1(\mathcal{M}(F)) = 2$.

Theorem 1.3. *Let C be a smooth curve of genus g and Clifford index c with $g \geq 2c+5$ which is neither hyperelliptic nor bielliptic. If a line bundle $\mathcal{M}$ computes the Clifford index of C with $(3c/2)+3 < \deg\mathcal{M} \leq g-1$, then $g = 2c+5$ and $|\mathcal{M}| = |F+F'|$ such that $|F|$ and $|F'|$ are penciles of degree $c+2$. Moreover, $\mathcal{M}(F)$ computes the Clifford index of C.*

We also show that the line bundle $\mathcal{M}$ is half canonical unless C is a $\frac{c+2}{2}$-fold covering of an elliptic curve in Proposition 2.5. In fact, such a pair $(C, \mathcal{M})$ can be found on a $K3$-surface whose Picard group is generated by a hyperplane section in $\mathbb{P}^r$.

Theorem 1.4. *Let X be a general $K3$ surface in $\mathbb{P}^r$ $(r \geq 3)$ whose Picard group is generated by a hyperplane section H with $\deg X = 2r-2$. Let C be a smooth irreducible curve on X contained in the linear system $|2H|$. Then*

1. *$g = 2c+5$ and there are pencils $|F|$, $|F'|$ of degree $c+2$ such that $|\mathcal{O}_C(1)| = |F+F'|$,*
2. *both $\mathcal{O}_C(1)$ and $\mathcal{O}_C(1)(F)$ compute the Clifford index of C,*
3. *$\mathcal{O}_C(1)$ is normally generated, but $\mathcal{O}_C(1)(F)$ is not.*

Consequently, for any r one can find couples $(C, \mathcal{L})$ such that $\mathcal{L}$ is an extremal line bundle on C with $h^1(\mathcal{L}) = 2$ and $h^0(\mathcal{L}) = r+1$. For a curve C of genus $g \equiv 1 (\mathrm{mod}\ 4)$, C has both a normally generated line bundle $\mathcal{M}$ and a non-normally generated line bundle $\mathcal{L}$ with $h^1(\mathcal{L}) = 2$ computing the Clifford index of C at the same time.

Recall the following: The Clifford index of the curve C is defined by

$$\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(\mathcal{L}) : h^0(\mathcal{L}) \geq 2,\ h^1(\mathcal{L}) \geq 2\}.$$

A line bundle $\mathcal{L}$ is said to compute the Clifford index of C if $\mathrm{Cliff}(\mathcal{L}) = \mathrm{Cliff}(C)$ with $h^0(\mathcal{L}) \geq 2$ and $h^1(\mathcal{L}) \geq 2$. The Clifford dimension of C is defined by

$$r(C) := \min\{h^0(\mathcal{L}) - 1 : \mathcal{L} \text{ computes the Clifford index of } C\}.$$

It has known that a general k-gonal curve has Clifford dimension 1 ([Ba86], [KeK89]). A smooth curve C is called an *exceptional* curve if $r(C) \geq 2$.

Notations

We denote the canonical line bundle on C by K, $H^i(C, \mathcal{L})$ by $H^i(\mathcal{L})$ and $H^i(C, \mathcal{O}(D))$ by $H^i(D)$. We abuse the notations as follows: $L \in |\mathcal{L}|$, $|\mathcal{O}(D)| = |D|$. For a divisor D on C, we denote $\langle D \rangle_{\mathcal{L}}$ the linear space spanned by D in the embedding associated to a very ample line bundle $\mathcal{L}$.

2. Line bundles computing the Clifford index

In this section, we have the following results: Let C be a curve of genus g and Clfford index c which is neither hyperelliptic nor bielliptic, and a line bundle $\mathcal{M}$ compute the Clifford index of C with $\deg\mathcal{M} \leq g-1,\ h^0(\mathcal{M}) \geq 4$. Then there is a quadric hypersurface of rank ≤ 4 containing $\varphi_{\mathcal{M}}(C)$ if and only if $|\mathcal{M}| = |g^1_{c+2} + h^1_{c+2}|$ and $g = 2c+5$. Thus if $g \geq 2c+5$ and $\deg\mathcal{M} > \frac{3c}{2} + 3$, then $|\mathcal{M}| = |g^1_{c+2} + h^1_{c+2}|$ and $g = 2c+5$. Moreover in this case, $\mathcal{M}$ is half-canonical unless C is a $\frac{c+2}{2}$-fold covering of an elliptic curve.

This is an extension of Corollary 3.2.5 in [CM91]. They showed that $\deg\mathcal{M} \leq \frac{3c}{2} + 3$ for a line bundle $\mathcal{M}$ computing the Clifford index of C if $g > 2c+4$ (resp. $g > 2c+5$) and c is odd (resp. even). In fact, the above is an extended result for the case $g = 2c+5$ and c is even. This is also comparable to the following: Let C be an exceptional curve of genus g and Clifford index c and a line bundle $\mathcal{M}$ compute the Clifford dimension of C. Then $\varphi_{\mathcal{M}}(C)$ is not contained any quadric

hypersurface of rank 4 or less if $r(C) \geq 3$. If deg$\mathcal{M} > (3c/2) + 3$, then $g = 2c + 4$ and $\mathcal{M}$ is half-canonical. Consult [ELMS89] for details .

Proposition 2.1 ([KKM90]). *Let C be a smooth curve of genus g and a line bundle $\mathcal{M}$ compute the Clifford index of C with* deg$\mathcal{M} \leq g - 1$ *and $h^0(\mathcal{M}) \geq 4$. Then $\mathcal{M}$ is birationallly very ample unless C is hyperelliptic or bielliptic.*

We note that the proposition also holds for such a linear system of any degree, since the condition $d \leq g - 1$ is not used in the proof of the proposition.

Lemma 2.2. *Let C be a smooth curve of genus g and a line bundle $\mathcal{M}$ compute the Clifford index of C with* deg$\mathcal{M} \leq g - 1$. *If C has a base point free linear system $|F|$ with $h^0(F) \geq 2$ and $h^0(\mathcal{M}(-F)) \geq 2$, then the linear system $|F|$, $|\mathcal{M}(-F)|$ and $|\mathcal{M}(F)|$ compute the Clifford index of C .*

Proof. We set deg$F = f$. Since $\mathcal{M}$ computes the Clifford index of C, we have Cliff$(\mathcal{M}(-F)) \geq$ Cliff$(\mathcal{M})$ and so $h^0(\mathcal{M}) \geq h^0(\mathcal{M}(-F)) + \frac{f}{2} \geq \frac{f}{2} + 2$ for $h^0(\mathcal{M}(-F)) \geq 2$. Assume $h^1(\mathcal{M}(F)) \leq 1$. Then by Rieman-Roch Theorem, $h^0(\mathcal{M}(F)) \leq f + 1$ for deg$\mathcal{M} \leq g - 1$. Thus $h^0(\mathcal{M}(F)) - h^0(\mathcal{M}) \leq (f + 1) - (\frac{f}{2} + 2) \leq \frac{f}{2} - 1$. Then by the base point free pencil trick([ACGH85], p 126), we get $\frac{f}{2} - 1 \geq h^0(\mathcal{M}(F)) - h^0(\mathcal{M}) \geq h^0(\mathcal{M}) - h^0(\mathcal{M}(-F)) \geq \frac{f}{2}$, which is a contradiction. Thus $h^1(\mathcal{M}(F)) \geq 2$ and so Cliff$(\mathcal{M}(F)) \geq$ Cliff$(\mathcal{M})$ which gives $h^0(\mathcal{M}(F)) - h^0(\mathcal{M}) \leq \frac{f}{2}$. Hence by the base point free pencil trick, we have $\frac{f}{2} \geq h^0(\mathcal{M}(F)) - h^0(\mathcal{M}) \geq h^0(\mathcal{M}) - h^0(\mathcal{M}(-F)) \geq \frac{f}{2}$. Consequently, Cliff$(\mathcal{M}(F)) =$ Cliff$(\mathcal{M}(-F)) =$ Cliff$(\mathcal{M}) =$ Cliff(C). Moreover all of them compute the Clifford index of C, since $h^1(\mathcal{M}(F)) \geq 2$ and deg$(\mathcal{M}(-F)) \leq g - 1$. Set $|\mathcal{M}(-F)| = |F'|$. Then $|F'|$ is base point free, since it computes the Clifford index of C. Hence by the same argument for $|F'|$ instead of $|F|$, the linear system $|F| = |\mathcal{M}(-F')|$ also computes the Clifford index of C. □

For divisors M and E on C, let (M, E) denote the greatest common divisor of them.

Lemma 2.3. *Let a line bundle $\mathcal{M}$ compute the Clifford index of C with* $\deg\mathcal{M} \leq g-1$, $h^0(\mathcal{M}) \geq 3$ *and $\mathcal{E}$ be a line bundle with $h^0(\mathcal{E}) \geq 2$. Then for any $P \in C$ there are divisors $M \in |\mathcal{M}|$ and $E \in |\mathcal{E}|$ such that $(M, E) = P$ or $P + Q$ for some $Q \in C$.*

Proof. Let P be an arbitrary point of C and E a divisor in $|\mathcal{E}|$ containing P. Set $E = P + \Sigma P_i$. Let B be the base locus of $|\mathcal{M}(-P)|$. Then B is either zero divisor or degree one divisor Q for some $Q \in C$, since $\mathcal{M}$ computes the Clifford index of C with $h^0(\mathcal{M}) \geq 3$. We set $\mathcal{G} := \mathcal{M}(-P-B)$. Then $\mathcal{G}$ is base point free of $h^0(\mathcal{G}) \geq 2$. Hence there is a $G \in |\mathcal{G}|$ such that $(G, \Sigma P_i) =$ zero divisor. Thus $(M, E) = P$ or $P + Q$ for $M = G + P + B \in |\mathcal{M}|$. $\square$

Using the above results, we get the following proposition.

Proposition 2.4. *Let C be a smooth curve of genus g and Clifford index c which is neither hyperelliptic nor bielliptic and let $\mathcal{M}$ be a line bundle computing the Clifford index of C with* $\deg\mathcal{M} \leq g-1$ *and $h^0(\mathcal{M}) \geq 4$. Then there is a quadric hypersurface of rank ≤ 4 containing $\varphi_{\mathcal{M}}(C)$ if and only if $|\mathcal{M}| = |F + F'|$ and $g = 2c+5$, where $|F|$ and $|F'|$ are pencils of degree $c+2$. In this case, $\mathcal{M}(F)$ computes the Clifford index of C.*

Proof. By the assumption and Proposition 2.1, the morphism $\varphi_{\mathcal{M}}$ is birational. Fix a quadric hypersurface Q of rank ≤ 4 containing $\varphi_{\mathcal{M}}(C)$. Let $|F_1|$ and $|F_2|$ be the complete linear systems induced by two pencils on Q. Then $|\mathcal{M}| = |F_1 + F_2|$. Let $|F|$ be a base point free pencil which is a subsystem of $|F_1|$. Then $h^0(\mathcal{M}(-F)) \geq h^0(F_2) \geq 2$. By Lemma 2.2, $|F| = g^1_{c+2}$, both $|\mathcal{M}(-F)|$ and $|\mathcal{M}(F)|$ also compute the Clifford index of C. In particular, we have $h^0(K \otimes \mathcal{M}(F)^{-1}) \geq 2$.
Claim: $|\mathcal{M}(-F)| = g^1_{c+2}$

To prove this, we assume $h^0(\mathcal{M}(-F)) \geq 3$. Let P be an arbitrary point of C. By Lemma 2.3, there are divisors $G_1 \in |\mathcal{M}(-F)|$ and $G_2 \in |K \otimes \mathcal{M}(F)^{-1}|$ such that $(G_1, G_2) = P$ or $P + R$ for some $R \in C$. Thus if we set $M = G_1 + F$ and $E = G_2 + F$, then $M \in |\mathcal{M}|$ and $E \in |K \otimes \mathcal{M}^{-1}|$ with $(M, E) = F + P$ or $F + P + R$. First we assume

$(M, E) = F + P$. Then by the sheaf exact sequence (see [Ha77], p 345)

$$0 \to \mathcal{O}((M, E)) \to \mathcal{O}(M) \oplus \mathcal{O}(E) \to \mathcal{O}(K(-(M, E))) \to 0,$$

we have $\mathrm{Cliff}(F + P) \le \mathrm{Cliff}(M)$. It is impossible because $\mathrm{Cliff}(M) = \mathrm{Cliff}(F) = c$.

As a consequence, $(M, E) = F + P + R$ for some $R \in C$. Using the above sheaf exact sequence, $\mathrm{Cliff}(F + P + R) = \mathrm{Cliff}(M) = \mathrm{Cliff}(F)$, and hence $h^0(K(-F - P - R)) = h^0(K(-F)) - 1$. On the other hand, we have $h^0(K(-F)) \ge h^0(\mathcal{M}) \ge 4$. Thus by Proposition 2.1, $|K(-F)|$ is birationally very ample since $\mathrm{Cliff}(K(-F)) = c$. Thus $h^0(K(-F-P-R)) = h^0(K(-F))-2$ except only finite pairs (P, R). It contradicts to the arbitrary choice of the point P. Thus $h^0(\mathcal{M}(-F)) = 2$ and so $|\mathcal{M}(-F)| = g^1_{c+2}$.

If we exchange the roles of $|\mathcal{M}(-F)|$ and $|K \otimes \mathcal{M}(F)^{-1}|$ in the claim, then we have $h^0(K \otimes \mathcal{M}(F)^{-1}) = 2$ and so $|K \otimes \mathcal{M}(F)^{-1}| = g^1_{c+2}$ since $K \otimes \mathcal{M}^{-1}$ also computes the Clifford index of C. Thus both $|\mathcal{M}|$ and $|K \otimes \mathcal{M}^{-1}|$ are sums of two linear pencils of degree $c+2$, which proves the result. $\square$

Proof of Theorem 1.3. The condition $(3c/2)+3 < \deg\mathcal{M}$ yields that $h^0(\mathcal{M}) \ge 3$. If $h^0(\mathcal{M}) = 3$, then we have $c = 1$ since $\mathcal{M}$ computes the Clifford index of C and $(3c/2)+3 < \deg\mathcal{M} = c+4$. Thus C is a plane quintic, which cannot occur since $g \ge 2c+5$. Accordingly, $h^0(\mathcal{M}) \ge 4$.

Denote $h^0(\mathcal{M}) = r + 1$. Suppose that the result does not hold, then by Proposition 2.4, the image curve $C' = \varphi_{\mathcal{M}}(C)$ is not contained any quadric hypersurface of rank≤ 4. Then by the sheaf exact sequence

$$0 \to \mathcal{I}_{C'}(2) \to \mathcal{O}_{\mathbb{P}^r}(2) \to \mathcal{O}_{C'}(2) \to 0,$$

we have $h^0(\mathcal{M}^2) \ge 4r - 2$. Hence $\mathrm{Cliff}(\mathcal{M}^2) \le 2c - 4r + 6$ and $h^1(\mathcal{M}^2) \ge 2$ since $\deg\mathcal{M} = c + 2r$ and $g \ge 2c + 5$. Accordingly, $\mathrm{Cliff}(\mathcal{M}^2) \ge c$ and so $4r \le c+6$. Thus we have $\deg\mathcal{M} = c+2r \le c+\frac{c}{2}+3$ which is a contradiction. $\square$

We can show that such a line bundle $\mathcal{M}$ is generally half-canonical for the boundary case $g = 2c + 5$. It is comparable to the line bundle

computing the Clifford dimension on an exceptional curve C of genus $g(C) = 2\mathrm{Cliff}(C) + 4$.

Proposition 2.5. *Let C and $\mathcal{M}$ be the same as Theorem 1.3. Assume there is a quadric hypersurface of rank ≤ 4 containing $\varphi_{\mathcal{M}}(C)$. Then $\mathcal{M}$ is half-canonical unless C is a $\frac{c+2}{2}$-fold covering of an elliptic curve.*

Proof. By the above theorem we may set $|\mathcal{M}| = |F + F_1|$ and $|K \otimes \mathcal{M}^{-1}| = |F + F_2|$ such that $|F|, |F_1|$ and $|F_2|$ are base point free pencils of degree $c + 2$. Assume that $\mathcal{M}$ is not half-canonical, i.e., $|F_1| \neq |F_2|$. Consider $\phi_{F_1} \times \phi_{F_2} : C \to \mathbb{P}^3$ and let C' be a smooth model of $\phi_{F_1} \times \phi_{F_2}(C)$. Then we have a morphism $\psi : C \to C'$ such that $\pi \circ \psi = \phi_{F_1} \times \phi_{F_2}$ where π is a normalization morphism from C' to $\phi_{F_1} \times \phi_{F_2}(C)$. Take a divisor G_i on C' such that $F_i = \psi^*(G_i)$ and let $m := \deg\psi$, then $|G_i|$'s are base point free pencils of degree $\frac{c+2}{m}$ on C' since $|F_i|$'s are base point free. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & C' \arrow[d, "\pi = \phi_{G_1} \times \phi_{G_2}"] \\
C \arrow[ur, "\psi"] \arrow[r, "\phi_{F_1} \times \phi_{F_2}"] & \phi_{F_1} \times \phi_{F_2}(C) \subset \mathbb{P}^3
\end{tikzcd}$$

If we note that $\phi_{G_1} \times \phi_{G_2}$ is birational then there exists $G_i \in |G_i|$ such that $(G_1, G_2) = Q$ for any $Q \in C'$. Fix $G_1 := Q_1 + \cdots + Q_{\frac{c+2}{m}}$ and let $R_i := \psi^*(Q_i)$, then $\psi^*(G_1) = R_1 + \cdots + R_{\frac{c+2}{m}} \in |F_1|$. Hence $F + \psi^*(G_1) \in |\mathcal{M}|$ and there exists $G_{2,i} \in |G_2|$ such that $(G_1, G_{2,i}) = Q_i$ for each i. Let $M := F + \psi^*(G_1)$ and $E_i := F + \psi^*(G_{2,i})$. Then $(M, E_i) = F + \psi^*(Q_i) = F + R_i$. Note that $\mathrm{Cliff}(F + R_i) = c$ for any i by the short exact sequence

$$0 \to \mathcal{O}(M, E_i) \to \mathcal{O}(M) \oplus \mathcal{O}(E_i) \to \mathcal{O}(K(-(M, E_i))) \to 0.$$

Let $s := \frac{c+2}{m}$.
claim : $\mathrm{Cliff}(F + R_1 + \cdots + R_k) = c$ for $k = 1, \ldots, s$.

First we prove that $\mathrm{Cliff}(F + R_1 + R_2) = c$. By the definition of Clifford index, we obtain $\mathrm{Cliff}(F + R_1 + R_2) = \deg(F + R_1 + R_2) -$

$2r(F+R_1+R_2) \geq c$. So, we get $\dim\langle F+R_1+R_2\rangle_K \geq c+m$ from the geometric Riemann-Roch Theorem.

On the other hand, $\dim\langle F+R_i\rangle_K = c+\frac{m}{2}$ and $\dim\langle F\rangle_K \cap \langle R_i\rangle_K = \frac{m}{2}-1$ since $\mathrm{Cliff}(F+R_i)=c$, $\dim\langle F\rangle_K = c$ and $\dim\langle R_i\rangle_K = m-1$. It produces $\dim\langle F+R_1+R_2\rangle_K \leq c+m$. Therefore, $\mathrm{Cliff}(F+R_1+R_2)=c$. In the same manner, one can prove that $\dim\langle F+R_1+\cdots+R_k\rangle_K = c+\frac{km}{2}$, which gives a proof of the claim.

If $s>2$, then we lead to $\deg(F+R_1+\cdots+R_{s-1}) > \frac{3c}{2}+3$, which gives a contradiction to Theorem 1.3. This contradiction gives us that $\mathcal{M}$ is half-canonical. If $s=2$, then there are different pencils $|G_1|$, $|G_2|$ of degree 2 on C'. Thus C' is an elliptic curve and hence C is a $\frac{c+2}{2}$-fold covering of the elliptic curve C'. We have excluded this case. In all, we obtain the result. $\square$

Proof of Theorem 1.4. Since X is a K3-surface in $\mathbb{P}^r$, $\mathcal{O}_C(2)$ is the canonical bundle of C, $g(C)=2c+5$ and $\deg\mathcal{O}_C(1)=2c+4$ from adjunction formula. By Green and Lazarsfeld Theorem ([GL87], Theorem), $\mathcal{O}_C(1)$ computes the Clifford index of C. Therefore the results (1) and (2) follow from Theorem 1.3.

X is projectively normal since a hyperplane section is a canonical curve. From the exact sequence:

$$0 \to H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{O}_{\mathbb{P}^r}(2)) \to H^0(\mathcal{O}_X(2)) \to 0,$$

one can see that $h^0(\mathcal{I}_X(2)) = \binom{r+2}{2} - \deg C - 2$ by the Riemann-Roch Theorem. We have $h^0(\mathcal{I}_C(2)) = \binom{r+2}{2} - \deg C - 1$ by the short exact sequence:

$$0 \to H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_X) \to 0.$$

The morphism $\mu_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$ is surjective for $m=2$ by $h^0(\mathcal{O}_C(2)) = g = \deg C + 1$. For $m \geq 3$, the surjectivity of μ_m follows from the proof of Theorem 3.6 in [ELMS89]. Therefore $\mathcal{O}_C(1)$ is normally generated.

Corollary 3.3 in the next section implies that $\mathcal{O}_C(1)(F)$ fails to be normally generated. $\square$

3. Extremal line bundles $\mathcal{L}$ with $h^1(\mathcal{L}) = 2$

We start this section to give a geometric interpretation of the surjectivity of Theorem 1.2.

Lemma 3.1. *Let $\mathcal{M}$ be a very ample line bundle on a smooth curve C, $\mathcal{F}$ a globally generated line bundle with $h^0(\mathcal{F}) = 2$ and $\mathcal{L} := \mathcal{M} \otimes \mathcal{F}$. For any two distinct divisors $F, F' \in \mathcal{F}$, $\langle F \rangle_{\mathcal{L}} \cap \langle F' \rangle_{\mathcal{L}} = \emptyset$ if and only if the cup product morphism $\mu : H^0(\mathcal{M}) \otimes H^0(F) \to H^0(\mathcal{L})$ is surjective.*

Proof. By the base point free pencil trick, we have the following exact sequence;

$$0 \to H^0(\mathcal{M}(-F)) \to H^0(\mathcal{M}) \otimes H^0(F) \xrightarrow{\mu} H^0(\mathcal{L}).$$

Let $r := h^0(\mathcal{M}) - 1$. We have $\dim\langle F \rangle_{\mathcal{L}} = h^0(\mathcal{L}) - r - 2$ and $\dim\langle F + F' \rangle_{\mathcal{L}} = h^0(\mathcal{L}) - h^0(\mathcal{M}(-F)) - 1$.

If μ is surjective, then $h^0(\mathcal{L}) = 2(r+1) - h^0(\mathcal{M}(-F))$. Therefore $\dim\langle F + F' \rangle_{\mathcal{L}} = 2\dim\langle F \rangle_{\mathcal{L}} + 1$ which implies $\langle F \rangle_{\mathcal{L}} \cap \langle F' \rangle_{\mathcal{L}'} = \emptyset$.

If $\langle F \rangle_{\mathcal{L}} \cap \langle F' \rangle_{\mathcal{L}} = \emptyset$, then $\dim\langle F + F' \rangle_{\mathcal{L}} = 2\dim\langle F \rangle_{\mathcal{L}} + 1$. This yields $h^0(\mathcal{L}) = 2(r+1) - h^0(\mathcal{M}(-F))$, so μ is surjective. □

Proof of Theorem 1.2. First, we claim that F fails to impose independent conditions on quadrics in $\langle F \rangle_{\mathcal{M}}$, i.e. $H^1(\mathcal{I}_{F/\langle F \rangle_{\mathcal{M}}}(2)) \neq 0$. Since $H^2(\mathcal{I}_{F/\mathbb{P}^m}(2)) = 0$, we have the following exact sequence:

$$H^1(\mathcal{I}_{F/\mathbb{P}^m}(2)) \to H^1(\mathcal{I}_{F/C}(2)) \to H^2(\mathcal{I}_{C/\mathbb{P}^m}(2)) \to 0$$

where $\mathbb{P}^m := \mathbb{P}(H^0(\mathcal{M}))$. Note that we have $H^2(\mathcal{I}_C(2)) \simeq H^1(\mathcal{O}_C(2)) \simeq H^1(\mathcal{M}^2)$ and $H^1(\mathcal{I}_{F/C}(2)) \simeq H^1(\mathcal{M}^2(-F))$. By the assumption (2), we conclude that $H^1(\mathcal{I}_{F/\mathbb{P}^m}(2)) \neq 0$. Since $H^1(\mathcal{I}_{F/\mathbb{P}^m}(2)) = H^1(\mathcal{I}_{F/\langle F \rangle_{\mathcal{M}}}(2))$, F fails to impose independent conditions on quadrics in $\langle F \rangle_{\mathcal{M}}$.

By Lemma 3.1 hypothesis (3) makes it possible to take two divisors $F, F' \in |g^1_{c+2}|$ such that $\langle F \rangle_{\mathcal{L}} \cap \langle F' \rangle_{\mathcal{L}} = \emptyset$ for $\mathcal{L} := \mathcal{M}(F)$. Consider the projection $\pi_{F'}$ of $\phi_{\mathcal{L}}(C)$ from F'. Then we have the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, "\phi_{\mathcal{L}}"] \arrow[dr, "\phi_{\mathcal{M}}"'] & \phi_{\mathcal{L}}(C) \subset \mathbb{P}^r (:= \mathbb{P}(H^0(\mathcal{L}))) \arrow[d, "\pi_{F'}\text{: projection}"] \\
& \phi_{\mathcal{M}}(C) \subset \mathbb{P}^m (:= \mathbb{P}(H^0(\mathcal{M})))
\end{tikzcd}$$

Since $\langle F\rangle_{\mathcal{L}} \cap \langle F'\rangle_{\mathcal{L}} = \emptyset$, we see that $H^1(\mathcal{I}_{F/\langle F\rangle_{\mathcal{M}}}(2)) \neq 0$ if and only if $H^1(\mathcal{I}_{F/\langle F\rangle_{\mathcal{L}}}(2)) \neq 0$. Therefore F fails to impose independent conditions on quadrics in $\langle F\rangle_{\mathcal{L}}$, i.e. $H^1(\mathcal{I}_{F/\langle F\rangle_{\mathcal{L}}}(2)) \neq 0$. This is equivalent to $H^1(\mathcal{I}_{F/\mathbb{P}^r}(2)) \neq 0$. From the following exact sequence

$$0 \to \mathcal{I}_{C/\mathbb{P}^r}(2) \to \mathcal{I}_{F/\mathbb{P}^r}(2) \to \mathcal{I}_{F/C}(2) \to 0,$$

one can see that $H^1(\mathcal{I}_{C/\mathbb{P}^r}(2)) \neq 0$ since $H^1(\mathcal{I}_{F/C}(2)) = H^1(\mathcal{L}^2(-F)) = H^1(\mathcal{M}^2(F)) = 0$ by the hypothesis (1). Thus $\mathcal{L}(:= \mathcal{M}(F))$ fails to be normally generated. $\square$

Remark 3.2. *If* $\deg\mathcal{L} \geq g+1$, *then the multiplication map*

$$\mu_m : H^0(\mathcal{L}) \otimes H^0(\mathcal{L}^m) \to H^0(\mathcal{L}^{m+1})$$

are surjective for $m \geq 2$ ([Gr84], *Theorem* (4.e.1)). *Therefore,* $\mathcal{L}$ *is normally generated if* $\mathcal{L}$ *satisfies* 2*-normality.*

Corollary 3.3. *Let* C, $\mathcal{M}$ *and* F *be the same as in Theorem 1.3. Then* $\mathcal{M}(F)$ *is an extremal line bundle with* $h^1(\mathcal{M}(F)) = 2$.

Proof. The hypothesis (1) of Theorem 1.2 holds trivially since $\deg\mathcal{M} = g-1$. From Lemma 2.2, one can see that $K \otimes \mathcal{M}^{-2}(F) = K \otimes \mathcal{M}^{-1}(-F')$. Thus $h^1(\mathcal{M}^2(-F)) = h^1(\mathcal{M}(F')) = 2$. Also we have $h^1(\mathcal{M}^2) \leq 1$ since $\deg\mathcal{M} = g-1$. Therefore the hypothesis (2) of the Theorem 1.2 is satisfied. Using the base point free pencil trick and the Riemann-Roch Theorem $\mu : H^0(\mathcal{M}) \otimes H^0(F) \to H^0(\mathcal{M}(F))$ is surjective, so the hypothesis (3) holds. Hence the result follows from Theorem 1.2 and Theorem 1.3. $\square$

4. Examples and questions

In this section, we observe examples of extremal line bundles $\mathcal{L}$ with $h^1(\mathcal{L}) \leq 2$.

Proposition 4.1 ([GL86], Remark 2.6)**.** *Let C be a k-gonal curve such that* Cliff(C) *is computed by a pencil g^1_k. Let $\sum_{i=1}^4 P_i$ be a divisor on C such that $h^0(g^1_k - P_i - P_j)) = 0$ for all $i, j \in \{1, \cdots, 4\}$. Then $\mathcal{L} := K(-g^1_k + \sum_{i=1}^4 P_i)$ is a nonspecial extremal line bundle on C.*

The following examples also can be found in [Ko02].

Proposition 4.2. *Let C be a nonsingular plane curve degree $d \geq 5$ and H a line section of C. Take a degree 4 divisor $Z \leq H$ and put $D = H - Z$. Then $K(-D)$ is an extremal line bundle on C with $h^1(K(-D)) = 1$.*

Proof. Since C has no pencil of degree $\leq d-2$, $K(-D)$ is very ample. By the Riemann-Roch theorem Z spans a line in $\mathbb{P}^r := \mathbb{P}(H^0(K(-D)))$, and hence fails to impose independent conditions on quadrics, i.e. $h^1(\mathcal{I}_{Z/\mathbb{P}^r}(2)) \neq 0$. Now consider the following exact sequence;

$$0 \to \mathcal{I}_{C/\mathbb{P}^r}(2) \to \mathcal{I}_{Z/\mathbb{P}^r}(2) \to \mathcal{I}_{Z/C}(2) \to 0$$

Since $H^1(\mathcal{I}_{Z/C}(2)) = H^1((K(-D))^2(-Z)) = 0$, $H^1(\mathcal{I}_{C/\mathbb{P}^r}(2)) \neq 0$, i.e. $K(-D)$ fails to be normally generated. Since Cliff$(K(-D)) = d - 4$, $K(-D)$ is an extremal line bundle. $\square$

Let C be an exceptional curve with $g = 2c + 4$ and $\mathcal{M}$ compute the Clifford dimension of C. Then we have deg$\mathcal{M} = 4r-3$, Cliff$(C) = 2r-3$ and $g = 4r - 2$ where $h^0(\mathcal{M}) = r + 1$. It is known that $\phi_{\mathcal{M}}(C)$ is projectively normal and has a $(2r-3)$-secant $(r-2)$-space divisor D ([CM91], Theorem A). Konno proved that the line bundle $K(-D)$ is extremal with $h^1(K(-D)) = 1$ in [Ko02]. We reprove it using Theorem 1.2.

Proposition 4.3. *Let $(C, \mathcal{M}, D)$ be as above. Then $K(-D)$ is an extremal line bundle with $h^1(K(-D)) = 1$.*

Proof. Let $\mathcal{F} \cong \mathcal{M}(-D)$, then $K(-D) \cong \mathcal{M} \otimes \mathcal{F}$ since $\mathcal{M}$ is a half-canonical. Conditions (1), (2) of Theorem 1.2 clearly hold. By the base point free pencil trick we have the following:

$$0 \to H^0(\mathcal{M} \otimes \mathcal{F}^{-1}) \to H^0(\mathcal{M}) \otimes H^0(\mathcal{F}) \xrightarrow{\mu} H^0(\mathcal{M} \otimes \mathcal{F}).$$

Then μ is surjective since $h^0(\mathcal{M}\otimes\mathcal{F})=h^0(K(-D))=c+4$, $h^0(\mathcal{M}\otimes\mathcal{F}^{-1})=1$ and $h^0(\mathcal{M})=\frac{c+5}{2}$. Theorem 1.2 implies that $K(-D)$ is an extremal line bundle since $\mathrm{Cliff}(K(-D))=\mathrm{Cliff}(C)$. $\square$

Even though the curves in next examples are lying on K3 surfaces, we do the works with explicit calculations.

Example 4.4. *Let C be a smooth complete intersection of smooth surfaces of degree 2 and 4 in $\mathbb{P}^3$. Then $\deg C=8$, $g(C)=9$, and $\mathrm{gon}(C)=4$. Therefore H. Martens' Theorem implies $\mathrm{Cliff}(C)=2$. For $\mathcal{M}:=\mathcal{O}_C(1)$, we have $|\mathcal{M}|=|F+F'|$ where F is a pencil of degree 4 by Theorem 1.3. Thus $\mathcal{M}(F)$ fails to be normally generated by Theorem 1.2. So $\mathcal{M}(F)$ is extremal with $h^1(\mathcal{M}(F))=2$.*

Example 4.5. *Let C be a smooth complete intersection of smooth hypersurfaces of degree $2,2,3$ in P^4. Then there is a quadric hypersurface Q of rank ≤ 4 and $\mathcal{M}(F)$ is an extremal line bundle with $h^1(\mathcal{M}(F))=2$. Here $\mathcal{M}=\mathcal{O}_C(1)$ and F is given by a ruling of Q of degree 6.*

Proof. Note that $g(C)=13$. By Lazarsfeld's theorem ([La97], Example 4.12), C can not have a pencil g^1_5. It implies that $\mathrm{Cliff}(C)\geq 3$ by Coppens and Martens' Theorem ([CM91]). Assume that $\mathrm{Cliff}(C)=3$. If the Clifford dimension $r(C)$ of C is 2, then C is isomorphic to a smooth plane septic, which cannot occur since $g(C)=13$. Therefore $r(C)\geq 3$ and there is a $g^{r(C)}_d$ with $d=3+2r(C)\leq 12=g(C)-1$, and so $r(C)\leq 4$. Hence C is an ELMS curve ([ELMS89] Section 5), which is a contradiction because of $g(C)\neq 2c+4$. Thus $\mathrm{Cliff}(C)=4$ since $|\mathcal{M}|=g^4_{12}$ where $\mathcal{M}:=\mathcal{O}_C(1)$. Note that C is always contained in a quadric of rank 4 or less because C is contained in two quadrics in $\mathbb{P}^4$. Whence Proposition 2.4 yields $|\mathcal{M}|=|F+F'|$ such that $|F|,|F'|$ are base point free pencils of degree $c+2$. By Theorem 1.2, $\mathcal{M}(F)$ fails to be normally generated. So $\mathcal{M}(F)$ is extremal with $h^1(\mathcal{M}(F))=2$. $\square$

Example 4.6. *Let $C\subset\mathbb{P}^5$ be a smooth complete intersection of four quadric hypersurfaces. Then there is a quadric hypersurface Q of rank ≤ 4 and $\mathcal{M}(F)$ is an extremal line bundle with $h^1(\mathcal{M}(F))=2$. Here $\mathcal{M}=\mathcal{O}_C(1)$ and F is given by a ruling of Q of degree 8.*

Proof. As in example 4.5, one can show that $g(C) = 17$ and $\mathrm{Cliff}(C) = 6$. Note that the quadrics of rank ≤ 4 in $\mathbb{P}^5$ form a closed subvariety of codimension 3 in the projective space of all quadrics in $\mathbb{P}^5$. Also, C is always contained in a quadric of rank ≤ 4 because C is contained in four quadrics. Therefore, by Proposition 2.4, $|\mathcal{M}| = |F + F'|$ for any hyperplane section $\mathcal{M}$. By Theorem 1.2, $\mathcal{M}(F)$ fails to be normally generated. So $\mathcal{M}(F)$ is extremal with $h^1(\mathcal{M}(F)) = 2$. $\square$

Finally we ask a couple of questions relating to the above results.

Question 4.7. *Is there an extremal line bundle $\mathcal{L}$ on a smooth curve with $h^1(\mathcal{L}) \geq 3$?*

Question 4.8. *Can we find a smooth curve which does not lie on a $K3$ surface, but has an extremal line bundle $\mathcal{L}$ with $h^1(\mathcal{L}) \geq 2$?*

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[1]Department of Mathematics, Seoul National University, Seoul 151-742, Korea

E-mail address: ychoi@math.kaist.ac.kr

[2]Department of Electronics, Chungwoon University, Chungnam, 350-701, Korea

E-mail address: sjkim@chungwoon.ac.kr

[3]Department of Mathematics, Konkuk University, Seoul, 143-701, Korea

E-mail address: rocky777@math.snu.ac.kr